Numerical Analysis of Quasiholes of the Moore-Read Wavefunction

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(Dated: January 22, 2009)

We demonstrate numerically that non-Abelian quasihole excitations of the $\nu = 5/2$ fractional quantum Hall state have some of the key properties necessary to support quantum computation. We find that as the quasihole spacing is increased, the unitary transformation which describes winding two quasiholes around each other converges exponentially to its asymptotic limit and that the two orthogonal wavefunctions describing a system with four quasiholes become exponentially degenerate. We calculate the length scales for these two decays to be $\xi_U \approx 2.7\,\ell_0$ and $\xi_E \approx 2.3\,\ell_0$ respectively. Additionally we determine which fusion channel is lower in energy when two quasiholes are brought close together.

arXiv:0901.3502v1 [cond-mat.mes-hall] 22 Jan 2009

The proposal to use quantum Hall states as a platform for quantum computation has spurred a great deal of interest[1]. These quantum Hall systems are believed to have natural "topological" immunity to decoherence and therefore hold particular promise for quantum computation. In so-called non-Abelian quantum Hall systems, the ground state is highly degenerate in the presence of quasiparticles, and this degenerate space can be used to store quantum information. Operations on this space are then performed by adiabatically dragging quasiparticles around each other, thus "braiding" their world-lines in 2+1 dimensions.

Although there is currently no definitive experimental evidence that non-Abelian quantum Hall states even exist, the community now strongly suspects[1] that the quantum Hall plateau observed at Landau level filling fraction $\nu = 5/2$ is the non-Abelian Moore-Read phase[2] (or its closely related particle-hole conjugate[3]). While the Moore-Read phase is, strictly speaking, not capable of universal topological quantum computation (computation by braiding quasiparticles around each other at large distances), a scheme has been devised[4] that in principle allows error free quantum computation by supplementing these topological processes with nontopological processes where quasiparticles are moved together and allowed to interact. Furthermore, the Moore-Read phase is frequently viewed as the simplest paradigm of a non-Abelian state of matter, and is therefore a logical starting point for detailed analysis[1].

In order for topological (or partially topological) schemes for quantum computation to be scalable (i.e., to allow large scale quantum computation), a number of crucial conditions must hold[1]. Condition (1) *As all of the quasiparticles are moved apart from one another, the splitting of the energy levels of the putatively degenerate ground state space must converge to zero at least as fast as e^{-R/ξ_E} where R is the minimum distance between quasiparticles.* In the literature, there has been numerical work suggesting that condition (1) may not be true[5] for the Moore-Read state. One of the goals of our work is to perform more precise numerical calculations to determine whether this numerical conclusion holds up to more careful scrutiny. Condition (2) *As quasiparticles are moved apart from each other, the unitary transformation that results from adiabatically dragging one quasiparticle around another must converge to its asymptotic limit at least as fast as e^{-R/ξ_U}.* For the Moore-Read state, this condition has analytically been shown to be true by Read[6], and here we will reconfirm this fact numerically. However, in the proof by Read[6], the precise length scale ξ_U remains unknown. Presumably ξ_E and ξ_U are both on the scale of a magnetic length multiplied by some number of order unity. If this number of "order unity" happens to be very large, it could in principle start to cause trouble for practical implementation of topological schemes. We will explicitly determine both ξ_U and ξ_E numerically. Finally, Condition (3) *One must be able to measure the topological quantum number associated with a group of quasiparticles.* Proposals have been made that such quantum numbers can be measured using interferometry[1,7]. However, this scheme has turned out to be very difficult experimentally. Another possible way to measure the topological quantum number of, say, two quasiparticles, is to move the quasiparticles microscopically close and precisely measure the force between them (or equivalently the energy change of moving them). While this may not sound any easier, it nonetheless proposes a different route to making this measurement should interferometry prove to be impossible. In the current paper we will attempt to numerically evaluate this energy change and show how it reflects the quantum number of a pair of quasiparticles.

Our numerical work is performed on a spherical geometry with a monopole of flux N_ϕ at the center of the sphere and N electrons on the surface. For the Moore-Read state[2], N_ϕ is given by $N_\phi = 2N - 3 + n_{qh}/2$ and n_{qh} is the number of quasiholes (quasiparticles with positive charge). The radius of the sphere is $(N_\phi/2)^{1/2}\ell_0$ where ℓ_0 is the magnetic length. For the the purpose of stating the decay lengths ξ_U and ξ_E, the distance R between quasiholes will be written in terms of the chord length. The definitions of ξ_E and ξ_U are given below. It should be noted that while their precise values depend on the the particular quasihole configurations used in our

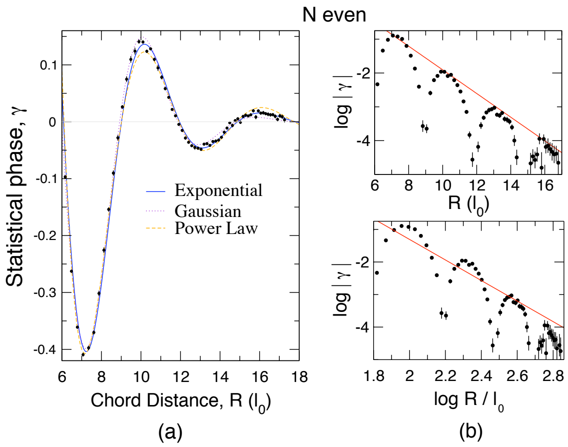

FIG. 1: (color online) Statistical phase for winding one quasihole around another in the spherical geometry as defined in the text. Data is shown for the case of even total numbers of electrons for which the topological quantum number of the pair of quasiholes is 1. In (a) we have plotted the statistical phase versus the chord distance R between the two quasiholes and fit the data using $\cos(a\frac{R}{l_0}+b)$ for the oscillatory part and a decay term that is either exponential (solid), Gaussian (dotted) or power law (dashed). We fit the data starting at $R = 6.48\, l_0$ and find the value of the reduced χ^2 is smallest for exponential decay with a value of 1.42 while for power law and Gaussian decays it is 7.22 and 5.52, respectively. The good fit to exponential decay is also confirmed when we plot the absolute value of the data on log and log-log scales (b) and perform linear fits to the extrema of the oscillations. The linear fit is clearly much better on the log plot demonstrating that the oscillations decay exponentially rather than as a power law.

calculations, alternate definitions for different quasihole configurations will give results that only differ by factors of order unity.

We consider the Moore-Read wavefunction in the presence of quasiholes which is defined as the zero energy space of a special short-ranged three-body interaction[8]. Although this is just a model interaction, the ground state wavefunction turns out to be an accurate approximation for more realistic interactions[9]. Pairs of quasihole excitations carry the topological quantum number "1" or "ψ" (sometimes known as "0" and "1/2"[13]) which represent the two states of a qubit, and the degeneracy[10] of a system with n_{qh} quasiholes is $2^{\frac{n_{qh}}{2}-1}$.

We start by considering the case of two quasiholes for which the ground state is unique. In this case we can address condition (2) above by calculating the braiding statistics of these two quasiholes. To do so in the spherical geometry we compute the Berry's phase accumulated when one quasihole is moved adiabatically around the equator of the sphere while the second quasihole is held fixed first on the north pole and then on the south pole. Both these Berry's phases have contributions from the statistical phase associated with the two quasiholes, and the Aharanov-Bohm phase due to the applied magnetic field. To isolate the statistical phase we therefore compute the *difference* between these two phases. In the planar geometry this difference would correspond to the change in Berry's phase when one quasihole in moved in a closed loop while a second quasihole is held fixed first inside the loop and then outside the loop.

Here the Berry's phases are all calculated numerically using a Monte-Carlo method essentially identical to that described in Ref. 14. (Details of the methods used will be given in Ref. 15). For the cases of either an even or odd number of electrons on the sphere, the two quasiholes together must have topological quantum numbers 1 or ψ, respectively. The statistical phase is then expected[1,2,6,13] to converge either to zero (if the quantum number is 1) or π (if the quantum number is ψ) as the distance between the quasiholes is increased. Indeed, in Figs. 1 and 2 we show that as the sphere is made larger, the convergence is exponential as required in both cases, and further we determine that, again in both cases, the decay scale is roughly $\xi_U \approx 2.7\, \ell_0$.

The difference between the even and odd case can be interpreted as the non-Abelian component – i.e, the part

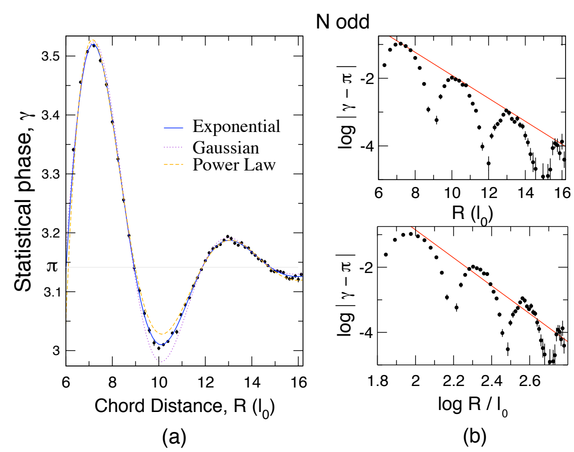

FIG. 2: (color online) Statistical phase for winding one quasihole around another in the same geometry as in Fig. 1, but now for the case of odd total numbers of electrons for which the two quasiholes have topological quantum number ψ. In the same manner as in Fig. 1 in (a) we plot the statistical phase versus the chord distance between the pair of quasiholes and perform a fit using the same functions for the oscillatory and decay parts (although in this case the oscillations are centered around the asymptotic value of π). We fit the data starting at $R = 6.32\, l_0$ and find the value of the reduced χ^2 is again smallest for exponential decay with a value of 0.97 while for power law and Gaussian decays χ^2 is 1.88 and 3.06, respectively. The good fit to exponential decay is once again confirmed when we plot the absolute value of the data on log and log-log scales (b) and perform linear fits to the extrema of the oscillations.

of the phase that depends on which topological sector the two quasiparticles are in. We conclude that this non-Abelian contribution does indeed converge exponentially with increasing system size as desired by condition (2). (Ideally we would like to determine the unitary transformation that occurs on this two dimensional ground state space when particles are braided around each other as in Ref. 14. However, we have found that it is currently numerically too demanding to demonstrate exponential convergence in this more complicated situation).

The oscillations in Figs. 1 and 2 (and in the later Figures) are not unexpected. In the closely related system of a p-wave paired superfluid, the oscillating form of the wavefunctions can be calculated explicitly[11]. However, in this quantum Hall system those results would only be qualitative.

To address condition (1) above, we now turn to the case of four quasiholes, and we will restrict ourselves to an even number of electrons. In this case, there are two putatively degenerate ground state wavefunctions[2,13]. We use a 1st excited Landau level coulomb interaction appropriate for $\nu = 5/2$ to describe the interaction between electrons as described in Ref. 16. Using a spherical geometry, we place four quasiholes on the corners of an equilateral tetrahedron and implement standard Monte-carlo procedures to evaluate the energy splitting between the two eigenstates of the interaction within the two dimensional ground state subspace (Details will be presented in Ref. 15). Results are presented in Fig. 3 as a function of system size, and indeed it appears that the two blocks become degenerate exponentially as the distance between quasiparticle increases as required by condition (2) with a decay length of $\xi_E \approx 2.3\,\ell_0$. This result appears to contradict results of Ref. 5 which claimed an algebraic rather than exponential decay. However, we note that in contrast to our work, in Ref. 5 the lowest Landau level interaction was used, so direct comparison of the two works is not possible. The numerical difficulty of collecting data is substantial, so admittedly our error bars are currently somewhat larger than desirable. However, we will continue to collect data and these results will almost certainly improve.

Finally we turn to the issue of measurement, condition (3). Here, we start with four quasiholes at the corners of a tetrahedron on a relatively large sphere ($N = 40$) where the two ground state wavefunctions are close to degenerate. We then move quasihole 2 close to 1 and observe the change in energy of the two wavefunctions. It turns out (and we will show in detail in Ref. 15) that if we choose to move the quasiholes together along an appropriately chosen path, then the conformal block wavefunctions defined in Ref. 13 diagonalize the interaction. These two conformal block wavefunctions, known as $|1\rangle$ and $|\psi\rangle$, are constructed such that the pair of quasiholes 1 and 2 have topological quantum number 1 and ψ respectively. The results of such a calculation are shown in Fig. 4. We see that the energy of moving two quasiholes together is always positive simply due to the coulomb

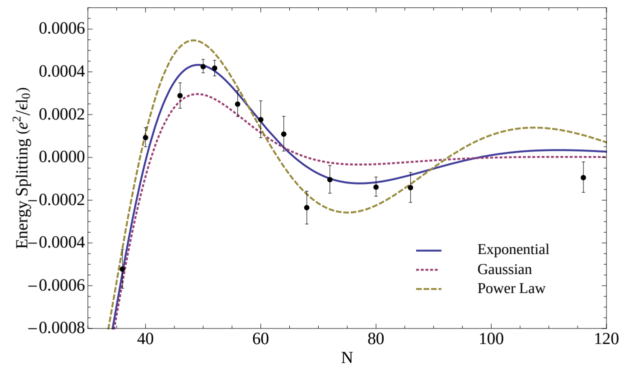

FIG. 3: (color online) Energy splitting of the eigenstates on a sphere with four quasiholes as a function of system size. The four quasiholes are placed on the corners of an equilateral tetrahedron. The data is fit with the function $\cos(a\sqrt{N}+b)$ multiplied by an exponential (solid), Gaussian (dotted), or power law (dashed) decay function. The distance between particles grows as $\sqrt{N}$, so this is essentially the same fit as used in Figs. 1 and 2. The data is fit starting with $N = 12$ (not shown because the fits are nearly identical at the small N values). The reduced χ^2 values for the exponential, Gaussian, and power law fits are 3.39, 7.73, and 6.49 respectively, which helps confirm our expectation that two energies become exponentially degenerate as the quasiholes move apart. We have not shown the log and log-log plots here because such plots discard the sign, and in the absence of a higher density of points, are hard to interpret.

repulsion. However, the energy is substantially greater when the two quasiholes are in the $|1\rangle$ state compared to the $|\psi\rangle$ state. To our knowledge, this result was not predicted. Thus our calculation makes the first mapping between a proposed measurement of the energy of two quasiholes and what this would indicate in terms of determining their topological quantum number. (Obviously if one were moving a quasiparticle together with a quasihole, the $|1\rangle$ state would have lower energy). We also point out that this result may be significant in regards to the possibility of quasiholes condensing into daughter states as proposed in Ref. 17.

The magnitude of the energy splitting of the two states is measured to be roughly $0.01e^2/\epsilon\ell_0$ which in a real system corresponds to roughly 1 K, a rather small energy to be measured. To make matters worse, this measured energy should be considered to be an upper bound, as mixing with states above the gap will be substantial and could easily reduce this energy scale (the experimentally measured gap itself is less than 1 K in the very best samples, although theoretically without disorder the gap could be almost 2.5 K. See Ref. 1 and therein). Nonetheless, this numerical work gives the first order of magnitude estimate for how large the splitting due to topological quantum numbers is likely to be compared to the overall coulomb energy between the two quasiholes.

To summarize, we have used Monte-Carlo techniques

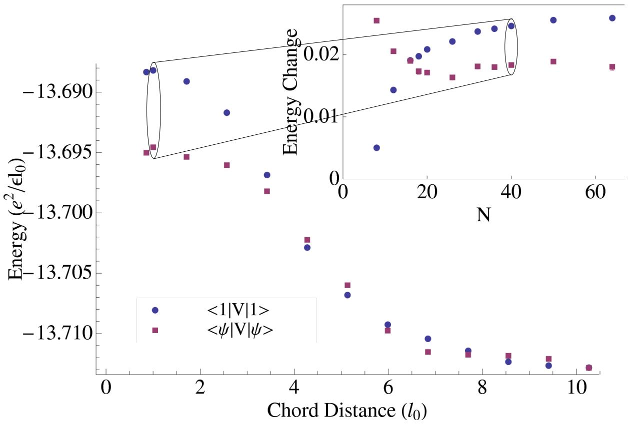

FIG. 4: (color online) This figure shows the total energy of the $N = 40$ system with four quasiholes as quasihole 2 is moved closer to its pair, quasihole 1. Quasihole 1 is located at the north pole and quasihole 2 is moved along a path that keeps a certain analytic form[13] for the trial states $|1\rangle$ and $|\psi\rangle$ precisely orthogonal[15]. The two states of the system, $|1\rangle$ and $|\psi\rangle$, which are nearly degenerate when the quasi-holes are well separated, split as the quasiholes approach each other. The inset shows the energy needed to move quasihole 2 within ℓ_0 of quasihole 1 for different values of N. We find that it takes more energy to bring the two quasiholes together when the system is in state $|1\rangle$ than in state $|\psi\rangle$, and that the energy splitting is on the order of $0.01e^2/\epsilon\ell_0$.

to examine several key properties of the Moore-Read wavefunction with quasiholes. Note that because our calculations consider wavefunctions projected to a single Landau level, they are equally applicable to the recently proposed AntiPfaffian wavefunction[3]. We find that both the unitary transformation associated with adiabatic transport and the energy splitting of putatively degenerate states converge exponentially with increasing distance between quasiparticles, and we explicitly extract the decay lengths. Encouragingly, the decay lengths are on the order of a magnetic length which suggests that quasiparticle spacing should not be a barrier to physical implementation of topological operations. Further we examine the energy splitting that occurs when two quasiholes are moved together. We find that the $|1\rangle$ state of these two particles is of *higher* energy and we measure this energy splitting between $|1\rangle$ and $|\psi\rangle$. Although this energy splitting is small, it gives experimentalists another way to measure topological quantum numbers in these systems. Many more details of this work will be presented in an upcoming publication[15].

Acknowledgments

We thank N. Read and S. M. Girvin for helpful input. M.B.'s computer time was supported by NSF grants DMR-0603369 and DMR-0653377. G.Z. and N.E.B. are supported by US DOE Grant No. DE-FG02-97ER45639.

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